Magnetic Field

A moving charged particle produces a magnetic field around itself.

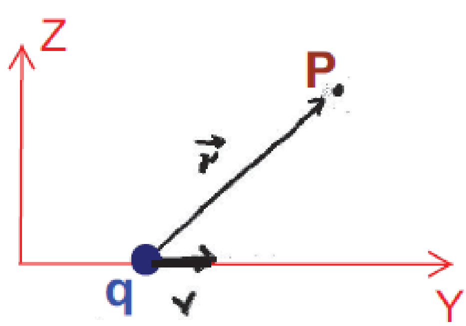
Thus a current of moving charged particles produces a magnetic field around the current.

This feature of *electromagnetism*, which is the combined study of electric and magnetic effects, came as a surprise to the people who discovered it.

This feature has become enormously important in everyday life because it is the basis of countless electromagnetic devices.

- Motors and generators, Transformers, Relays , Electric bells and buzzers
- Loudspeakers and headphones, Magnetic recording and data storage equipment: tape recorders, VCRs, hard disks
- MRI machines, Scientific equipment such as mass spectrometers
- Particle accelerators, Magnetic locks, Magnetic separation equipment

Magnetic field produced by a moving point charge



Magnetic field at point P

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{q\,\vec{v} \times \vec{r}}{r^3}$$

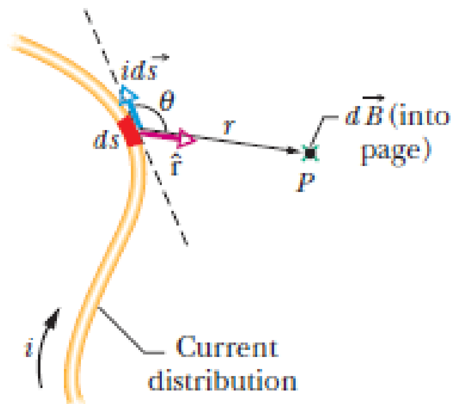
$$\frac{\mu_0}{4\pi} = 10^{-7}\ \frac{T\ m}{Amp.}$$

q = electrical charge (coulomb)
V = velocity (m/s)
r = vector position (m)
B = magnetic field (Tesla)

Magnetic field produced by current:

Figure shows a wire of arbitrary shape carrying a current *i*.

We want to find the magnetic field ***B*** at a nearby point *P*.



Divide the wire into differential elements *ds* and then define for each element a length vector ***ds*** that has length *ds* and whose direction is the direction of the current in *ds*.

We wish to calculate the field produced at *P* by a typical current-length element.

Thus, we can calculate the net field *B* at *P* by summing, via integration, the contributions dB from all the current-length elements.

However, this summation is more challenging than the process associated with electric fields because of a complexity; whereas a charge element dq producing an electric field is a scalar, a current-length element *ids* producing a magnetic field is a vector, being the product of a scalar and a vector.

The magnitude of the field ***dB*** produced at point ***P*** at distance ***r*** by a current length element ***ids*** turns out to be

$$dB = \frac{\mu_0}{4\pi} \frac{i\, ds \sin\theta}{r^2} \qquad \textbf{1}$$

where θ is the angle between the directions of ***ds*** and ***r***, a unit vector that points from *ds* toward *P*. μ_o is called the *permeability constant,* whose value is defined to be exactly

$$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{T \cdot m/A} \approx 1.26 \times 10^{-6}\mathrm{T \cdot m/A} \qquad \textbf{2}$$

The direction of ***dB*** , shown as being into the page in Fig. , is that of the cross product ***ds*** × ***r***

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{i\, d\vec{s} \times \hat{r}}{r^2} \qquad \textbf{3}$$

This vector equation and its scalar form, eq. 1, are known as the **law of Biot and Savart.**

The law, which is experimentally deduced, is an inverse-square law.

We shall use this law to calculate the net magnetic field produced at a point by various distributions of current.

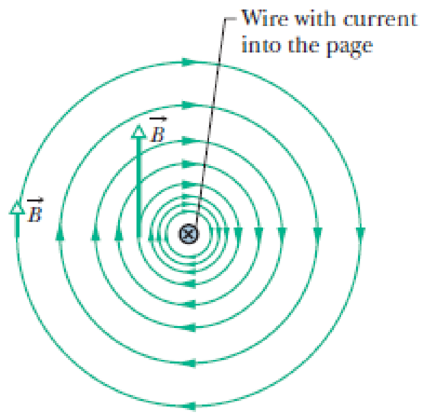
Magnetic Field Due to a Current in a Long Straight Wire

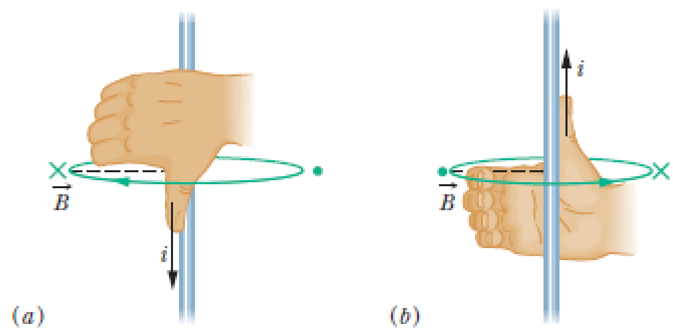
Using the law of Biot and Savart to find the magnitude of the magnetic field at a perpendicular distance *R* from a long (infinite) straight wire carrying a current *i* is given by

$$B = \frac{\mu_0 i}{2\pi R} \qquad \textbf{4}$$

The magnitude *B* depends only on the current and the perpendicular distance *R* of the point from the wire.

The field lines form concentric circles around the wire, as Fig. 2 and as the iron filings in Fig. 3. The increase in the spacing of the lines increasing distance from the wire represents the 1/*R* decrease in the magnitude of predicted by Eq. 2.The lengths of the two vectors in the figure also show the 1/*R* decrease. (Print page 14- 37)





Proof of above Equation

The wire is straight and of infinite length, illustrates the task at hand.

The magnetic field B at point *P*, a perpendicular distance *R* from the wire.

The magnitude of the differential magnetic field produced at *P* by the current-length element *i ds* located a distance *r* from *P* is given by :

$$dB = \frac{\mu_0}{4\pi} \frac{i\, ds \sin\theta}{r^2} \qquad (1)$$

The direction of d_B in Fig. is that of the vector $ds \times \hat{r}$ namely, directly into the page.

d_B at point *P* has this same direction for all the current-length elements into which the wire can be divided.

Thus, we can find the magnitude of the magnetic field produced at *P* by the current-length elements in the upper half of the infinitely long wire by integrating *dB* in Eq. 1

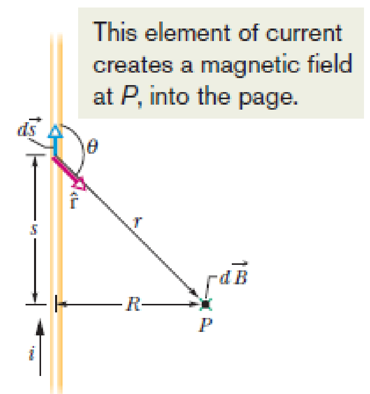


Fig. 29-5 Calculating the magnetic field produced by a current *i* in a long straight wire. The field $d\vec{B}$ at *P* associated with the current-length element $i\, d\vec{s}$ is directed into the page, as shown.

Now consider a current-length element in the lower half of the wire, one that is as far below *P* as d_s is above *P*. By Eq. 2, the magnetic field produced at *P* by this current-length element has the same magnitude and direction as that from element *i ds* in Fig. above.

Further, the magnetic field produced by the lower half of the wire is exactly the same as that produced by the upper half.

To find the magnitude of the total magnetic field *B* at *P*, we need only multiply the result of our integration by 2.

$$B = 2\int_0^\infty dB = \frac{\mu_0 i}{2\pi} \int_0^\infty \frac{\sin\theta\, ds}{r^2}$$

The variables θ, *s*, and *r* in this equation are not independent; by using above Fig. shows that they are related by

$$r = \sqrt{s^2 + R^2}$$

$$\sin\theta = \sin(\pi - \theta) = \frac{R}{\sqrt{s^2 + R^2}}.$$

By using this integral 19 $\quad$ **19.** $\displaystyle\int \frac{dx}{(x^2 + a^2)^{3/2}} = \frac{x}{a^2(x^2 + a^2)^{1/2}}$

Eq 2 becomes

$$B = \frac{\mu_0 i}{2\pi}\int_0^\infty \frac{R\,ds}{(s^2 + R^2)^{3/2}}$$

$$= \frac{\mu_0 i}{2\pi R}\left[\frac{s}{(s^2 + R^2)^{1/2}}\right]_0^\infty = \frac{\mu_0 i}{2\pi R} \qquad \textbf{6}$$

Note that the magnetic field at *P* due to either the lower half or the upper half of the infinite wire in above Fig. is half this value; that is,

$$B = \frac{\mu_0 i}{4\pi R} \qquad \textbf{7}$$

Magnetic Field Due to a Current in a Circular Arc of Wire

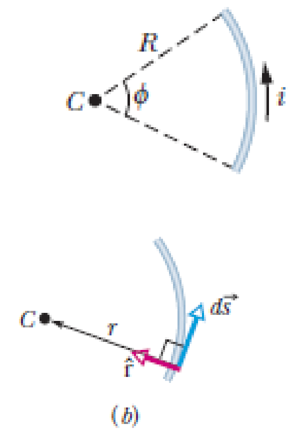


Figure 29-6*a* shows such an arc-shaped wire with central angle ϕ,

radius *R*, and center *C*, carrying current *i*.

At *C*, each current-length element *i ds* of the wire

Produces a magnetic field of magnitude *dB* given by Eq. 1.

Moreover, as Fig. 29-6*b* shows, no matter where the element is

located on the wire, the angle θ between the vectors and is 90°; also, $r = R$.

Thus, by substituting *R* for *r* and 90° for θ in Eq. 29-1, we obtain

$$dB = \frac{\mu_0}{4\pi}\frac{i\,ds\sin 90^\circ}{R^2} = \frac{\mu_0}{4\pi}\frac{i\,ds}{R^2} \qquad \textbf{8}$$

Magnitude of the field at *C*

An application of the right-hand rule anywhere along the wire (as in Fig. 29-6c) will show that all the differential fields have the same direction at C - directly out of the page.

Thus, the total field at C is simply the sum (via integration) of all the differential fields d_B.

We use the identity ds = R dϕ to change the variable of integration from ds to dϕ and obtain, from above Eq.

$$B = \int dB = \int_0^\phi \frac{\mu_0}{4\pi}\frac{iR\,d\phi}{R^2} = \frac{\mu_0 i}{4\pi R}\int_0^\phi d\phi$$

By integration

$$B = \frac{\mu_0 i\phi}{4\pi R} \qquad \textbf{9}$$

Note that this equation gives us the magnetic field *only* at the center of curvature of a circular arc of current.

When you insert data into the equation, you must be careful to express ϕ in radians rather than degrees.

For example, to find the magnitude of the magnetic field at the center of a full circle of current, you would substitute 2π rad for ϕ in above Eq. , finding

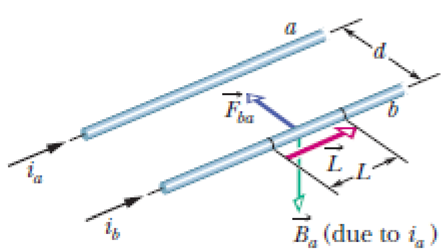
$$B = \frac{\mu_0 i(2\pi)}{4\pi R} = \frac{\mu_0 i}{2R} \qquad \textbf{10}$$

Force between Two Parallel Currents

Two long parallel wires carrying currents exert forces on each other.

Figure shows two such wires, separated by a distance *d* and carrying currents i_a and i_b.

Let us analyze the forces on these wires due to each other.



Two parallel wires carrying currents in the same direction attract each other. B_a is the magnetic field at wire *b* produced by the current in wire *a*. F_{ba} is the resulting force acting on wire *b* because it carries current in B_a.

First find the force on wire *b* in above Fig. due to the current in wire *a*.

That current produces a magnetic field B_a, and it is this magnetic field that actually causes the force we seek.

To find the force, then, we need the magnitude and direction of the field *at the site of wire b*.

The magnitude of B_a at every point of wire *b* is,

$$B_a = \frac{\mu_0 i_a}{2\pi d}. \qquad \mathbf{11}$$

The curled - straight right-hand rule tells us that the direction of B_a at wire *b* is down, as Fig. above,.

Now that we have the field, we can find the force it produces on wire *b*.

The force on a length *L* of wire *b* due to the external magnetic field is

$$\vec{F}_{ba} = i_b \vec{L} \times \vec{B}_a, \qquad \mathbf{12}$$

where *L* is the length vector of the wire. In Fig., vectors *L* and B_a are perpendicular to each other, and so, we can write by putting the value of B_a

$$F_{ba} = i_b L B_a \sin 90° = \frac{\mu_0 L i_a i_b}{2\pi d}. \qquad \mathbf{13}$$

The direction of F_{ba} is the direction of the cross product $L \times B_a$.

Applying the right-hand rule for cross products to *L* and B_a in Fig., we see that F_{ba} is directly toward wire *a*, as shown.

Note that

To find the force on a current-carrying wire due to a second current-carrying wire, first find the field due to the second wire at the site of the first wire. Then find the force on the first wire due to that field.

Ampere’s Law

We can find the net electric field due to *any* distribution of charges by first writing the differential electric field *dE* due to a charge element and then summing the contributions of from all the elements. However, if the distribution is complicated, we may have to use a computer. Recall, however, that if the distribution has planar, cylindrical, or spherical symmetry, we can apply Gauss’ law to find the net electric field with considerably less effort.

Similarly, we can find the net magnetic field *dB* due to *any* distribution of currents by first writing the differential magnetic field (Eq. 3) due to a current-length element and then summing the contributions of *dB* from all the elements. Again we may have to use a computer for a complicated distribution. However, if the distribution has some symmetry, we may be able to apply **Ampere’s law** to find the magnetic field with considerably less effort. This law, which can be derived from the Biot–Savart law, has traditionally been credited to André-Marie Ampère (1775–1836), for whom the SI unit of current is named. However, the law actually was advanced by English physicist James Clerk Maxwell.

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{enc} \qquad \text{(Ampere's law).} \qquad 14$$

The loop on the integral sign means that the scalar (dot) product $\vec{B} \cdot d\vec{s}$ is to be integrated around a closed loop, called an Amperian loop. The current i_{enc} is the net current encircled by that closed loop

To see the meaning of the scalar product $\vec{B} \cdot d\vec{s}$ and its integral, let us first apply Ampere’s law to the general situation of Fig. 11.The figure shows cross sections of three long straight wires that carry currents i_1, i_2, and i_3 either directly into or directly out of the page. An arbitrary Amperian loop lying in the plane of the page encircles two of the currents but not the third. The counterclockwise direction marked on the loop indicates the arbitrarily chosen direction of integration for Eq. 14.

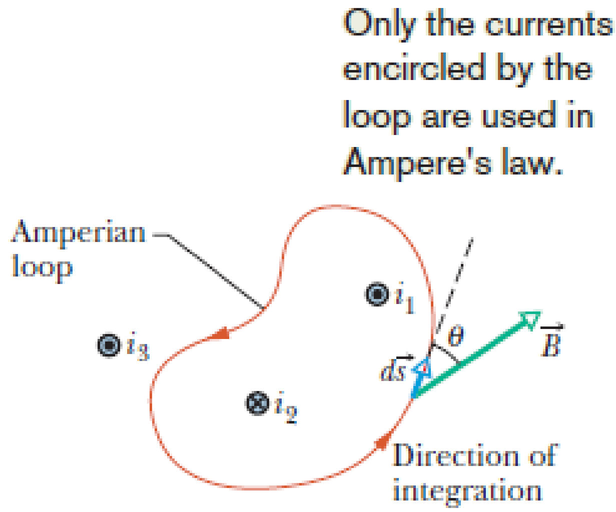


Fig. 29-11 Ampere's law applied to an arbitrary Amperian loop that encircles twc long straight wires but excludes a third wire. Note the directions of the currents.

To apply Ampere's law, we mentally divide the loop into differential vector elements ***ds***: that are everywhere directed along the tangent to the loop in the direction of integration. Assume that at the location of the element ***ds*** shown in Fig. 11, the net magnetic field due to the three currents is ***B***. Because the wires are perpendicular to the page, we know that the magnetic field at ***ds*** due to each current is in the plane of Fig. 11; thus, their net magnetic field ***B*** at ***ds*** must also be in that plane. However, we do not know the orientation of within the plane. In Fig. 11, ***B*** is arbitrarily drawn at an angle u to the direction of ***ds***.

The scalar product $\vec{B} \cdot d\vec{s}$ on the left side of Eq. 14 is equal to ***B* cos θ *ds***. Thus, Ampere's law can be written as

$$\oint \vec{B} \cdot d\vec{s} = \oint B \cos\theta \, ds = \mu_0 i_{enc}. \qquad 15$$

We can now interpret the scalar product $\vec{B} \cdot d\vec{s}$ as being the product of a length *ds* of the Amperian loop and the field component *B* cos u tangent to the loop. Then we can interpret the integration as being the summation of all such products around the entire loop.

When we can actually perform this integration, we do not need to know the direction of ***B*** before integrating. Instead, we arbitrarily assume ***B*** to be generally in the direction of integration (as in Fig. 11). Then we use the following curled–straight right-hand rule to assign a plus sign or a minus sign to each of the currents that make up the net encircled current i_{enc}:

Curl your right hand around the Amperian loop, with the fingers pointing in the direction of integration. A current through the loop in the general direction of your outstretched thumb is assigned a plus sign, and a current generally in the opposite direction is assigned a minus sign.

Finally, we solve Eq. 15 for the magnitude of ***B***. If ***B*** turns out positive, then the direction we assumed for ***B*** is correct. If it turns out negative, we neglect the minus sign and redraw in the opposite direction.

In Fig. 12 we apply the curled - straight right-hand rule for Ampere's law to the situation of Fig. -11. With the indicated counterclockwise direction of integration, the net current encircled by the loop is

$$i_{enc} = i_1 - i_2.$$

(Current i_3 is not encircled by the loop.) We can then rewrite Eq. 15 as

$$\oint B \cos \theta \, ds = \mu_0 (i_1 - i_2). \qquad 16$$

You might wonder why, since current $\boldsymbol{i_3}$ contributes to the magnetic-field magnitude ***B*** on the left side of Eq. 6, it is not needed on the right side. The answer is that the contributions of current $\boldsymbol{i_3}$

to the magnetic field cancel out because the integration in Eq. 16 is made around the full loop. In contrast, the contributions of an encircled current to the magnetic field do not cancel out.

We cannot solve Eq. 16 for the magnitude ***B*** of the magnetic field because for the situation of Fig. 11 we do not have enough information to simplify and solve the integral. However, we do know the outcome of the integration; it must be equal to $\mu_\circ(i_1 - i_2)$, the value of which is set by the net current passing through the loop.

We shall now apply Ampere's law to two situations in which symmetry does allow us to simplify and solve the integral, hence to find the magnetic field.

Magnetic Field Outside a Long Straight Wire with Current

Figure 13 shows a long straight wire that carries current *i* directly out of the page. Equation 4 tells us that the magnetic field produced by the current has the same magnitude at all points that are the same distance *r* from the wire; that is, the field ***B*** has cylindrical symmetry about the wire. We can take advantage of that symmetry to simplify the integral in Ampere's law (Eqs. 14 and 15) if we encircle the wire with a concentric circular Amperian loop of radius *r*, as in Fig. 13. The magnetic field ***B*** then has the same magnitude ***B*** at every point on the loop. We shall integrate counterclockwise, so that has the direction shown in Fig. 13.

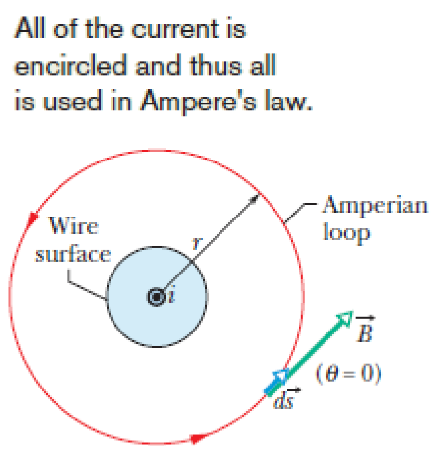


Fig. 13 Using Ampere's law to find the magnetic field that a current *i* produces outside a long straight wire of circular cross section. The Amperian loop is a concentric circle that lies outside the wire.

We can further simplify the quantity $B \cos \theta$ in Eq. 15 by noting that is tangent to the loop at every point along the loop, as is ***ds***. Thus, ***B*** and ***ds*** are either parallel or antiparallel at each point of the loop, and we shall arbitrarily assume the former. Then at every point the angle θ between ***ds*** and ***B*** is 0°, so $\cos \theta = \cos 0° = 1$.The integral in Eq. 15 then becomes

$$\oint \vec{B} \cdot d\vec{s} = \oint B \cos \theta \, ds = B \oint ds = B(2\pi r).$$

Note that $\oint ds$ is the summation of all the line segment lengths *ds* around the circular loop; that is, it simply gives the circumference $2\pi r$ of the loop.

Our right-hand rule gives us a plus sign for the current of Fig. 13.The right side of Ampere's law becomes $+\mu_\circ i$, and we then have

$$B(2\pi r) = \mu_0 i$$

$$B = \frac{\mu_0 i}{2\pi r} \quad \text{(outside straight wire).} \qquad 17$$

With a slight change in notation, this is Eq. 4, which we derived earlier – with considerably more effort - using the law of Biot and Savart. In addition, because the magnitude *B* turned out positive, we know that the correct direction of must be the one shown in Fig. 13.

Magnetic Field Inside a Long Straight Wire with Current

Figure 14 shows the cross section of a long straight wire of radius R that carries a uniformly distributed current i directly out of the page. Because the current is uniformly distributed over a cross section of the wire, the magnetic field $\boldsymbol{B}$ produced by the current must be cylindrically symmetrical.

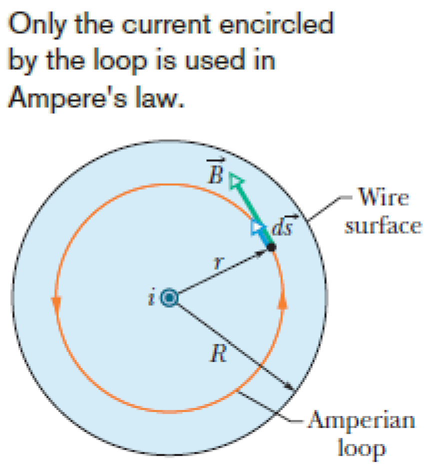


Fig. 14 Using Ampere's law to find the magnetic field that a current i produces inside a long straight wire of circular cross section. The current is uniformly distributed over the cross section of the wire and emerges from the page. An Amperian loop is drawn inside the wire.

Thus, to find the magnetic field at points inside the wire, we can again use an Amperian loop of radius r, as shown in Fig. 14, where now $r < R$. Symmetry again suggests that $\boldsymbol{B}$ is tangent to the loop, as shown; so the left side of Ampere's law again yields

$$\oint \vec{B} \cdot d\vec{s} = B \oint ds = B(2\pi r). \qquad 18$$

To find the right side of Ampere's law, we note that because the current is uniformly distributed, the current i_{enc} encircled by the loop is proportional to the area encircled by the loop; that is,

$$i_{enc} = i\frac{\pi r^2}{\pi R^2}. \qquad 19$$

Our right-hand rule tells us that i_{enc} gets a plus sign. Then Ampere's law gives us

$$B(2\pi r) = \mu_0 i\frac{\pi r^2}{\pi R^2}$$

$$B = \left(\frac{\mu_0 i}{2\pi R^2}\right)r \quad \text{(inside straight wire).} \qquad 20$$

Thus, inside the wire, the magnitude ***B*** of the magnetic field is proportional to *r* , is zero at the center, and is maximum at $r = R$ (the surface). Note that Eqs. 17 and 20 give the same value for *B* at the surface.

Solenoids and Toroids

Magnetic Field of a Solenoid

We now turn our attention to another situation in which Ampere's law proves useful. It concerns the magnetic field produced by the current in a long, tightly wound helical coil of wire. Such a coil is called a **solenoid** (Fig. 16).We assume that the length of the solenoid is much greater than the diameter.

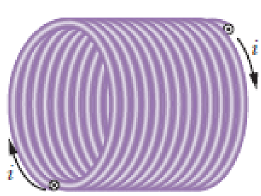


Fig. 16 A solenoid carrying current *i*.

Figure 17 shows a section through a portion of a "stretched-out" solenoid.

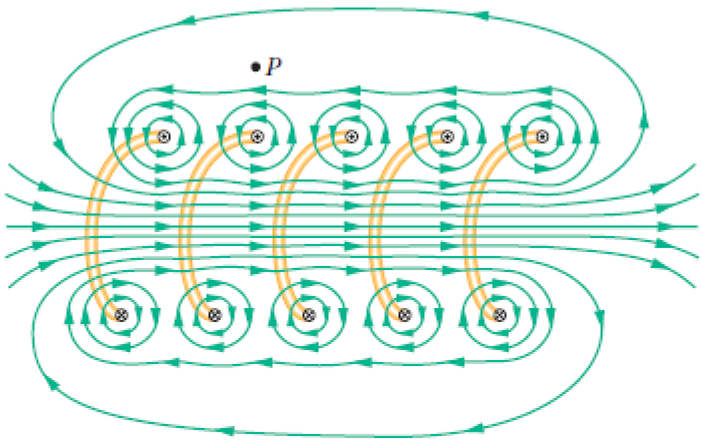


Fig. 17 A vertical cross section through the central axis of a "stretched-out" solenoid. The back portions of five turns are shown, as are the magnetic field lines due to a current through the solenoid. Each turn produces circular magnetic field lines near itself. Near the solenoid's axis, the field lines combine into a net magnetic field that is directed along the axis. The closely spaced field lines there indicate a strong magnetic field. Outside the solenoid the field lines are widely spaced; the field there is very weak.

The solenoid's magnetic field is the vector sum of the fields produced by the individual turns (windings) that make up the solenoid. For points very close to a turn, the wire behaves magnetically almost like a long straight wire, and the lines of ***B*** there are almost concentric circles. Figure 17 suggests that the field tends to cancel between adjacent turns. It also suggests that, at points inside the solenoid and reasonably far from the wire, ***B*** is approximately parallel to the (central) solenoid axis. In the limiting case of an ideal solenoid, which is infinitely long and consists of tightly packed (close-packed) turns of square wire, the field inside the coil is uniform and parallel to the solenoid axis.

At points above the solenoid, such as *P* in Fig. 17, the magnetic field set up by the upper parts of the solenoid turns (these upper turns are marked ⊙) is directed to the left (as drawn near *P*) and tends to cancel the field set up at *P* by the lower parts of the turns (these lower turns are marked

$\otimes$), which is directed to the right (not drawn). In the limiting case of an ideal solenoid, the magnetic field outside the solenoid is zero. Taking the external field to be zero is an excellent assumption for a real solenoid if its length is much greater than its diameter and if we consider external points such as point *P* that are not at either end of the solenoid. The direction of the magnetic field along the solenoid axis is given by a curled – straight right-hand rule: Grasp the solenoid with your right hand so that your fingers follow the direction of the current in the windings; your extended right thumb then points in the direction of the axial magnetic field.
Figure 18 shows the lines of for a real solenoid. The spacing of these lines in the central region shows that the field inside the coil is fairly strong and uniform over the cross section of the coil. The external field, however, is relatively weak.
Let us now apply Ampere's law,

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{\text{enc}}, \qquad 21$$

to the ideal solenoid of Fig. 19, where ***B*** is uniform within the solenoid and zero outside it, using the rectangular Amperian loop abcda. We write $\oint \vec{B} \cdot d\vec{s}$ as the sum of four integrals, one for each loop segment:

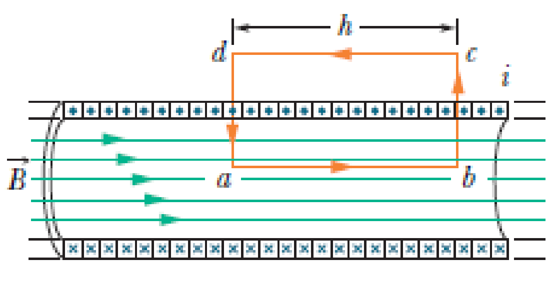


Fig. 19 Application of Ampere's law to a section of a long ideal solenoid carrying a current *i*. The Amperian loop is the rectangle *abcda*.

$$\oint \vec{B}\cdot d\vec{s} = \int_a^b \vec{B}\cdot d\vec{s} + \int_b^c \vec{B}\cdot d\vec{s} + \int_c^d \vec{B}\cdot d\vec{s} + \int_d^a \vec{B}\cdot d\vec{s}. \qquad 22$$

The first integral on the right of Eq. 29-22 is ***Bh,*** where ***B*** is the magnitude of the uniform field ***B*** inside the solenoid and ***h*** is the (arbitrary) length of the segment from *a* to *b*. The second and fourth integrals are zero because for every element ***ds*** of these segments, either is perpendicular to ***ds*** or is zero, and thus the product $\vec{B}\cdot d\vec{s}$ is zero. The third integral, which is taken along a segment that lies outside the solenoid, is zero because *B*= 0 at all external points. Thus, $\oint \vec{B}\cdot d\vec{s}$ for the entire rectangular loop has the value **Bh.**

The net current i_{enc} encircled by the rectangular Amperian loop in Fig. 19 is not the same as the current ***i*** in the solenoid windings because the windings pass more than once through this loop. Let ***n*** be the number of turns per unit length of the solenoid; then the loop encloses ***nh*** turns and

$$i_{\text{enc}} = i(nh).$$

Ampere's law then gives us

$$Bh = \mu_0 inh$$

$$B = \mu_0 in \qquad \text{(ideal solenoid).} \qquad 23$$

Although we derived Eq. 23 for an infinitely long ideal solenoid, it holds quite well for actual solenoids if we apply it only at interior points and well away from the solenoid ends. Equation 23 is consistent with the experimental fact that the magnetic field magnitude *B* within a solenoid does not depend on the diameter or the length of the solenoid and that *B* is uniform over the solenoidal cross section. A solenoid thus provides a practical way to set up a known uniform magnetic field for experimentation, just as a parallel-plate capacitor provides a practical way to set up a known uniform electric field.

Magnetic Field of a Toroid

Figure 20*a* shows a toroid, which we may describe as a (hollow) solenoid that has been curved until its two ends meet, forming a sort of hollow bracelet. What magnetic field is set up inside the toroid (inside the hollow of the bracelet)? We can find out from Ampere's law and the symmetry of the bracelet.

From the symmetry, we see that the lines of form concentric circles inside the toroid, directed as shown in Fig. 29-20*b*.

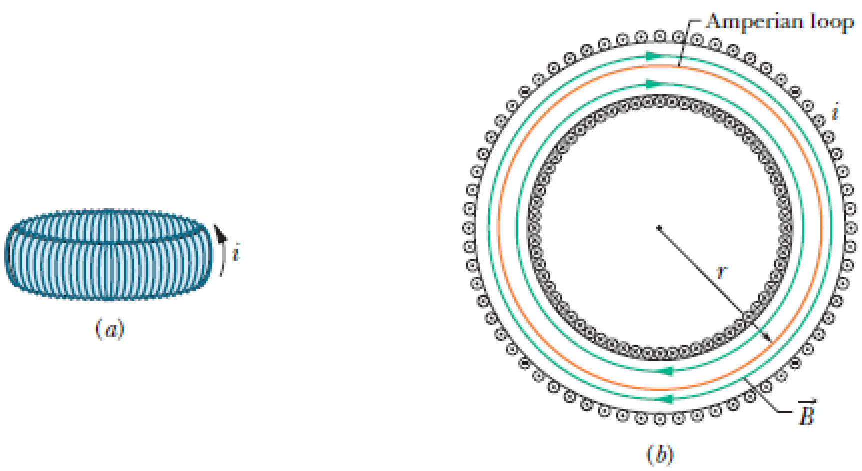


Fig. 20 (*a*) A toroid carrying a current *i*. (*b*) A horizontal cross section of the toroid. The interior magnetic field (inside the bracelet-shaped tube) can be found by applying Ampere's law with the Amperian loop shown.

Let us choose a concentric circle of radius *r* as an Amperian loop and traverse it in the clockwise direction.Ampere's law (Eq. 14) yields

$$(B)(2\pi r) = \mu_0 i N$$

where ***i*** is the current in the toroid windings (and is positive for those windings enclosed by the Amperian loop) and ***N*** is the total number of turns. This gives

$$B = \frac{\mu_0 i N}{2\pi}\frac{1}{r} \quad \text{(toroid).} \qquad 24$$

In contrast to the situation for a solenoid, *B* is not constant over the cross section of a toroid.
It is easy to show, with Ampere's law, that *B* = 0 for points outside an ideal toroid (as if the toroid were made from an ideal solenoid). The direction of the magnetic field within a toroid follows from our curled - straight right-hand rule: Grasp the toroid with the fingers of your right hand curled in the direction of the current in the windings; your extended right thumb points in the direction of the magnetic field.

A Current-Carrying Coil as a Magnetic Dipole

So far we have examined the magnetic fields produced by current in a long straight wire, a solenoid, and a toroid. We turn our attention here to the field produced by a coil carrying a current. We saw that such a coil behaves as a magnetic dipole in that, if we place it in an external magnetic field ***B*** , a torque given by

$$\vec{\tau} = \vec{\mu} \times \vec{B} \qquad 25$$

acts on it. Here ***μ*** is the magnetic dipole moment of the coil and has the magnitude ***NiA***, where ***N*** is the number of turns, ***i*** is the current in each turn, and ***A*** is the area enclosed by each turn. (*Caution:* Don't confuse the magnetic dipole moment with the permeability constant.)
Recall that the direction of ***μ*** is given by a curled–straight right-hand rule: Grasp the coil so that the fingers of your right hand curl around it in the direction of the current; your extended thumb then points in the direction of the dipole moment ***μ*** .

Magnetic Field of a Coil

We turn now to the other aspect of a current-carrying coil as a magnetic dipole. What magnetic field does *it* produce at a point in the surrounding space? The problem does not have enough symmetry to make Ampere's law useful; so we must turn to the law of Biot and Savart. For simplicity, we first consider only a coil with a single circular loop and only points on its perpendicular central axis, which we take to be a *z* axis. We shall show that the magnitude of the magnetic field at such points is

$$B(z) = \frac{\mu_0 i R^2}{2(R^2 + z^2)^{3/2}}, \qquad 26$$

in which *R* is the radius of the circular loop and ***z*** is the distance of the point in question from the center of the loop. Furthermore, the direction of the magnetic field ***B*** is the same as the direction of the magnetic dipole moment $\vec{\mu}$ of the loop.

For axial points far from the loop, we have $z >> R$ in Eq. 26. With that approximation, the equation reduces to

$$B(z) \approx \frac{\mu_0 i R^2}{2z^3}$$

Recalling that πR^2 is the area *A* of the loop and extending our result to include a coil of *N* turns, we can write this equation as

$$B(z) = \frac{\mu_0}{2\pi} \frac{NiA}{z^3}.$$

Further, because ***B*** and ***μ*** have the same direction, we can write the equation in vector form, substituting from the identity ***μ = NiA:***

$$\vec{B}(z) = \frac{\mu_0}{2\pi} \frac{\vec{\mu}}{z^3} \qquad \text{(current-carrying coil).} \qquad 27$$

Thus, we have two ways in which we can regard a current-carrying coil as a magnetic dipole:

(1) it experiences a torque when we place it in an external magnetic field;

(2) it generates its own intrinsic magnetic field, given, for distant points along its axis, by Eq. 27. Figure 21 shows the magnetic field of a current loop; one side of the loop acts as a north pole (in the direction of) and the other side as a south pole, as suggested by the lightly drawn magnet in the figure. If we were to place a current-carrying coil in an external magnetic field, it would tend to rotate just like a bar magnet would.

Proof of Equation

$$B(z) = \frac{\mu_0 i R^2}{2(R^2 + z^2)^{3/2}}$$

Figure 22 shows the back half of a circular loop of radius ***R*** carrying a current ***i***. Consider a point ***P*** on the central axis of the loop, a distance ***z*** from its plane. Let us apply the law of Biot and Savart to a differential element ***ds*** of the loop, located at the left side of the loop. The length vector ***ds*** for this element points perpendicularly out of the page. The angle θ between ***ds*** and $\hat{r}$ in Fig. 22 is 90°; the plane formed by these two vectors is perpendicular to the plane of the page and contains both $\hat{r}$ and $d\vec{s}$. From the law of Biot and Savart (and the right-hand rule), the differential field ***dB*** produced at point ***P*** by the current in this element is perpendicular to this plane and thus is directed in the plane of the figure, perpendicular to , as indicated in Fig. 22.

Let us resolve $d\vec{B}$ into two components: $dB_{\parallel}$, along the axis of the loop and $dB_{\perp}$ perpendicular to this axis. From the symmetry, the vector sum of all the perpendicular components $dB_\perp$ due to all the loop elements *ds* is zero. This leaves only the axial (parallel) components $dB_\parallel$, and we have

$$B = \int dB_\parallel .$$

For the element ***ds*** in Fig. 22, the law of Biot and Savart (Eq. 1) tells us that the magnetic field at distance *r* is

$$dB = \frac{\mu_0}{4\pi} \frac{i\, ds \sin 90^\circ}{r^2}.$$

We also have

$$dB_\parallel = dB \cos \alpha.$$

Combining these two relations, we obtain

$$dB_\parallel = \frac{\mu_0 i \cos \alpha\, ds}{4\pi r^2}. \qquad 28$$

Figure 22 shows that ***r*** and a are related to each other. Let us express each in terms of the variable *z*, the distance between point *P* and the center of the loop. The relations are

$$r = \sqrt{R^2 + z^2} \qquad 29$$

and

$$\cos \alpha = \frac{R}{r} = \frac{R}{\sqrt{R^2 + z^2}}. \qquad 30$$

Substituting Eqs. 29-29 and 29-30 into Eq. 29-28, we find

$$dB_\parallel = \frac{\mu_0 i R}{4\pi (R^2 + z^2)^{3/2}}\, ds.$$

Note that ***i***, ***R***, and ***z*** have the same values for all elements ***ds*** around the loop; so when we integrate this equation, we find that

$$B = \int dB_{\parallel}$$

$$= \frac{\mu_0 i R}{4\pi(R^2 + z^2)^{3/2}} \int ds$$

or, because $\int ds$ is simply the circumference $2\pi R$ of the loop,

$$B(z) = \frac{\mu_0 i R^2}{2(R^2 + z^2)^{3/2}}.$$